PMM U.S.S.R.,Vol.51,No.1,pp.1-5,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

ON THE STABILITY OF POINTS OF LIBRATION OF AN INHOMOGENEOUS TRIAXIAL ELLIPSOID*

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A uniformly rotating inhomogeneous gravitating ellipsoid is considered. On every ellipsoidal surface the density is constant. A material particle can move freely in the space filled by the gravitating mass. Existence conditions for relative equilibria are stated, and their stability is studied. The results are interpreted from the view-point of stellar system dynamics.

1. The existence of equilibria. The gravitating ellipsoid has semi-axes a_1, a_2, a_3 and rotates with constant angular velocity ω about one axis (a_3). Unlike /1/, it is assumed to be inhomogeneous. The density is distributed ellipsoidally, being constant on every similar ellipsoidal surface. The origin of the coordinate system x_i $(i = 1, 2, 3)$, which rotates with the ellipsoid, is located at its centre, while the coordinate axes are along the semi-axes a_i $(i = 1, 2, 3)$ respectively.

Taking $l = (a_1^2 + a_2^2 + a_3^2)^{1/2}$ and $T = 1/\omega$ as the characteristic dimension and time, we change to new (dimensionless) variables by the relations $x_i = l\xi_i$ $(i = 1, 2, 3)$, $t = T\tau$. The equations of motion of a passive particle have the Hamiltonian form

$$\xi^\cdot = H_\eta, \quad \eta^\cdot = -H_\xi; \quad \xi, \eta \in \mathbf{R}^3 \tag{1.1}$$

$$H = \frac{1}{2}\|\eta\|^2 + \eta_1\xi_2 - \eta_2\xi_1 - \Lambda(\xi), \quad \Lambda(\xi) = \rho\int_0^\infty h[\mu(u, \xi)]\frac{du}{Q(u)}$$

$$\mu(u, \xi) = \sum_{k=1}^3 \frac{\xi_k^2}{\alpha_k + u}, \quad \rho = \frac{3fm}{4\omega^2 l^3}, \quad h(\mu) = \int_\mu^\infty \gamma(v)\,dv$$

$$\int_0^\infty \gamma(u^{2/3})\,du = 1$$

$$\gamma(\mu) = \frac{4\pi a_1a_2a_3}{3m}\delta_1(\mu), \quad Q(u) = [(\alpha_1 + u)(\alpha_2 + u)(\alpha_3 + u)]^{1/2}$$

$$\delta_1[\mu(0, \xi)] = \delta(l\xi), \quad \|\eta\| = (\eta_1^2 + \eta_2^2 + \eta_3^2)^{1/2}, \quad a_i^2 = l\alpha_i$$

$$(i = 1, 2, 3)$$

The function δ_1 characterizes the density on an ellipsoidal layer, f is the gravitational constant, and m is the total mass of the ellipsoid. As a passive particle we can imagine a star inside an elliptic galaxy.

We find the equilibrium positions from the conditions: $H_\xi = H_\eta = 0$.

In more detail:

$$H_{\xi_1} = -\eta_2 - \Lambda_{\xi_1} = 0, \; H_{\xi_2} = \eta_1 - \Lambda_{\xi_2} = 0, \quad H_{\xi_3} = -\Lambda_{\xi_3} = 0$$

$$H_{\eta_1} = \eta_1 + \xi_2 = 0, \; H_{\eta_2} = \eta_2 - \xi_1 = 0, \quad H_{\eta_3} = \eta_3 = 0$$

In configuration space we obtain the system of equations

$$\xi_1 + \Lambda_{\xi 1} = \xi_2 + \Lambda_{\xi 2} = \Lambda_{\xi 3} = 0$$

If a weak condition such as $\gamma \in L_1(\mathbf{R}_+)$ is imposed on the mass distribution, we have the inclusion /2/ $\Lambda \in C^1(\mathbf{R}^3 \setminus \{0\})$. We can therefore calculate explicitly the derivatives $\Lambda_{\xi i} = -2\rho\xi_i F_i(\xi)$, where

$$F_i(\xi) = \int_{\mathbf{R}_+} \frac{\gamma[\mu(u, \xi)]}{(\alpha_i + u)Q(u)}\,du \quad (i = 1, 2, 3) \tag{1.2}$$

The equations of equilibrium take the form

$$\xi_i(1 - 2\rho F_i(\xi)) = 0 \;\; (i = 1, 2), \;\; \xi_3\rho F_3(\xi) = 0 \tag{1.3}$$

We shall consider various cases when these equations are solvable.

If $\xi_3^\circ \neq 0$, then $F_3(\xi^\circ) = 0$. Since the integrand is non-negative, we must have

*Prikl.Matem.Mekhan.,51,1,3-8,1987

$\gamma[\mu(u, \xi^\circ)] \equiv 0$ for $u \in \mathbf{R}_+$ almost everywhere. Hence $\gamma(\mu) = 0$ almost everywhere for $\mu \in [0, \mu(0, \xi)]$. This means that the ellipsoid $\mu(0, \xi) = \mu(0, \xi^\circ)$ is a bounded domain in $\mathbf{R}^3$, filled by a vacuum. This domain cannot be extended without limit, since the total mass of the ellipsoid is certainly non-zero.

Inside the domain $\gamma[\mu(u, \xi)] = 0$, so that $F_i(\xi) = 0$. From this and (1.3) we have $\xi_1 = \xi_2 = 0$, i.e., the equilibria cannot be isolated but must fill a piece of the axis of revolution lying in a cavity.

In stellar dynamics, ellipsoidal objects with a vacuum in the central domain are of no interest. We shall therefore consider henceforth points of libration which do not lie on the axis of revolution. From (1.3) we immediately obtain $\xi_3^\circ = 0$, and all the singular points, if there are such, lie in the equatorial plane.

If $1 - 2\rho F_i(\xi^\circ) \neq 0$, then $\xi_i = 0$ $(i = 1, 2, 3)$ must be the centre of the ellipsoid.

There remains the possibility $1 - 2\rho F_i(\xi^\circ) = 0$. Then, $1 - 2\rho F_j(\xi^\circ) \neq 0$ $(j \neq i)$, if $\alpha_1 \neq \alpha_2$. Otherwise we must have $\alpha_1 = \alpha_2$ and the ellipsoid is a spheroid, while the equilibria fill a circle.

In any case it suffices to confine ourselves to $i = 1$. By symmetry, the case $i = 2$ is considered in the same way. If $1 - 2\rho F_1(\xi^\circ) = 0$, then from (1.3) we must have $\xi_2^\circ = 0$. Hence the equilibrium position is on the ξ_1 axis and the problem of finding the points of libration reduces to seeking the solutions of the equation

$$f(\xi_1) \equiv 1 - 2\rho F_1(\xi_1, 0, 0) = 0 \tag{1.4}$$

Whether there are roots, and if so, how many, depends on the function γ as well as on the parameter values. In the case of actual elliptic galaxies, γ can be taken to be a continuous monotonically decreasing function.

Theorem 1. If the density $\gamma \in L_1(\mathbf{R}_+)$ is defined everywhere in $\mathbf{R}_+$ and does not increase as the argument increases, the necessary conditions for the existence of points of libration are

$$1 - 2\rho\gamma(0+) A_1 \leqslant 0$$

$$\gamma(0+) = \lim_{\mu \to 0+} \gamma(\mu), \quad A_i = \int_0^\infty \frac{du}{(\alpha_i + u) Q(u)} \quad (i = 1, 2, 3)$$

Here we have the alternative:

1°. Equilibrium exists and is unique only when

$$1 - 2\rho\gamma(0+) A_1 < 0$$

2°. If the equilibrium is not unique, it is not isolated. The solutions of Eq. (1.4) fill a piece of the ξ_1 axis, adjacent to the origin, while the density is constant on the family of ellipsoidal surfaces which pass through the piece.

Proof. At zero, the function f has the right-hand limit $f(\xi_1) \to 1 - 2\rho\gamma(0+) A_1$, $\xi_1 \to 0+$.

For, inasmuch as γ is defined and monotonic in $\mathbf{R}_+$, it has a right-hand limit at zero. Hence, given any sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that, for $|\xi_1^2/\alpha_1| < \delta$ we have $\gamma(0+) - \varepsilon < \gamma(\xi_1^2/\alpha_1) < \gamma(0+)$. Since γ is monotonic, for $u \in \mathbf{R}_+$ we also have $\gamma(0+) - \varepsilon < \gamma[\xi_1^2/(\alpha_1 + u)] \leqslant \gamma(0+)$, whence we obtain $[\gamma(0+) - \varepsilon] A_1 < F_1(\xi_1, 0, 0) \leqslant \gamma(0+) A_1$. Hence the limit $F_1(\xi_1, 0, 0) \to \gamma(0+) A_1$, $\xi_1 \to 0+$ exists.

Again, since γ is monotonic, f is also monotonic. It follows from /2/ that $f(\xi_1)$ is continuous for $\xi_1 \in (0, +\infty)$. Moreover, $f(\xi_1) \to 1$, $\xi_1 \to +\infty$.

For, γ satisfies the condition, given among relations (1.1), for the mass to be finite. For points on the ξ_1 axis, this condition can be written as

$$\frac{3}{2} \xi_1^3 \int_0^\infty \frac{\gamma[\xi_1^2/(\alpha_1 + u)]}{(\alpha_1 + u)^{5/2}} du = \int_0^a \gamma(v^{2/3}) dv \leqslant c_1 < +\infty, \quad a = \frac{\xi_1^3}{\alpha_1^{3/2}},$$

where $c_1 > 0$ is a constant. Since, as $u \to +\infty$, the limit $(\alpha_2 + u)(\alpha_3 + u)/(\alpha_1 + u)^2 \to 1$ exists, we can choose a constant c_2 such that $[(\alpha_2 + u)(\alpha_3 + u)]^{-1/2} \leqslant c_2/(\alpha_1 + u)$ for all $u \in \mathbf{R}_+$.

Thus we finally have $|F_1(\xi_1, 0, 0)| \leqslant 2c_1c_2/(3\xi_1^3) \to 0$, $\xi_1 \to +\infty$.

In short, if there is equilibrium at $\xi_1^\circ > 0$, then we must have $f(0+) \leqslant 0$. The first part of the theorem is proved.

Further, if the non-zero solution of (1.4) is unique, we must have $f(0+) < 0$. For, if $f(0+) = 0$, then by the monotonicity we have $f(\xi_1) = 0$ for all $\xi_1 \in (0, \xi_1^\circ)$, which contradicts the uniqueness.

Conversely, let $f(0+) < 0$ and at the same time let the uniqueness be violated, i.e., with $\xi_1', \xi_1'' > 0$, $\xi_1' < \xi_1''$, we have $f(\xi_1') = f(\xi_1'') = 0$. Then, $F_1(\xi_1'', 0, 0) - F_1(\xi_1', 0, 0) = 0$. Since $\gamma[(\xi_1'')^2/(\alpha_1 + u)] \leqslant \gamma[(\xi_1')^2/(\alpha_1 + u)]$, because γ is monotonic, then for all $u \in \mathbf{R}_+$ we must have

$$\gamma[(\xi_1'')^2/(\alpha_1 + u)] \equiv \gamma[(\xi_1')^2/(\alpha_1 + u)] \tag{1.5}$$

It can be shown that $\gamma(\mu) \equiv \text{const}$ for $\mu \leqslant (\xi_1'')^2/\alpha_1$.

We define u_1 from the equation $(\xi_1'')^2/(\alpha_1 + u_1) = (\xi_1')^2/\alpha_1$. We obtain $u_1 = \alpha_1 [(\xi_1''/\xi_1')^2 - 1] = \alpha > 0$.

We assume inductively that, for $u_k\ (k \leqslant n)$ which satisfy the equation

$$(\xi_1'')^2 \Big(\alpha_1 + \sum_{i=1}^{k-1} u_i + u_k\Big)^{-1} = (\xi_1')^2 \Big(\alpha_1 + \sum_{i=1}^{k-1} u_i\Big)^{-1}$$

we have $u_k \geqslant \alpha$. This is also true for u_{n+1}, since

$$u_{n+1} = \Big(\alpha_1 + \sum_{k=1}^{n} u_k\Big)[(\xi_1''/\xi_1')^2 - 1] \geqslant \alpha$$

Hence the series $u_1 + u_2 + \ldots$ is divergent, so that it can be shown that γ is constant. For, $\gamma[(\xi_1'')^2/(\alpha_1 + u_1)] = \gamma[(\xi_1')^2/\alpha_1] = \gamma[(\xi_1'')^2/\alpha_1]$, and since γ is monotonic, throughout the interval $[0, u_1]$, with $u \in [0, u_1]$, we have

$$\gamma[(\xi_1'')^2/(\alpha_1 + u)] = \gamma[(\xi_1'')^2/\alpha_1] = \text{const}$$

Using identity (1.5), the function on the left-hand side of (1.5) can be continued continuously from $[0, u_1]$ into $[0, u_1 + u_2]$, and so on, into any interval $[0, u_1 + u_2 + \ldots + u_n]$. Since the upper limit is the partial sum of a divergent series, continuation is admissible into the entire semi-infinite interval $[0, +\infty)$.

Hence $\gamma(\mu) = \text{const} = \gamma(0+)$ for $\mu \in (0, (\xi_1'')^2/\alpha_1]$. Hence $f(\xi_1'') = f(0+) = 0$, which contradicts our assumption. Consequently, there must only be one point of libration.

It is also clear from our proof that, when the uniqueness is violated, the equilibrium positions fill the interval $[0, (\xi_1'')^2/\alpha_1]$ of the ξ_1 axis, while the density is constant in the relevant family of ellipsoids. The theorem is proved.

2. Stability. By analogy with /1/, we pass to local coordinates by the canonical transformation $(\boldsymbol{\xi}, \boldsymbol{\eta}) \mapsto (\mathbf{q}, \mathbf{p})$ with the aid of the generating function $W(\boldsymbol{\xi}, \mathbf{p}) = (\boldsymbol{\xi} - \boldsymbol{\xi}^\circ, \mathbf{p} + \boldsymbol{\eta}^\circ)$, where $(\boldsymbol{\xi}^\circ, \boldsymbol{\eta}^\circ) = (\xi_1^\circ, 0, 0, 0, \xi_1^\circ, 0)$ is the position of equilibrium of the Hamiltonian system, if it exists, corresponding to the point of libration. The transformation relations are $\mathbf{q} = W_{\mathbf{p}} = \boldsymbol{\xi} - \boldsymbol{\xi}^\circ$, $\boldsymbol{\eta} = W_{\boldsymbol{\xi}} = \mathbf{p} + \boldsymbol{\eta}^\circ$.

The Hamiltonian function can be expanded in a power series in the neighbourhood of the equilibrium position (if the density has the required smoothness)

$$H(\mathbf{q}, \mathbf{p}) = \sum_{k=2}^{\infty} H_k(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\|\mathbf{p}\|^2 + p_1 q_2 - p_2 q_1 - \sum_{k=2}^{\infty} \Lambda_k(\mathbf{q}),$$

where H_k are homogeneous forms of the phase variables of degree k, and Λ_k are homogeneous forms of the coordinates in the expansion of the potential in the neighbourhood of $\boldsymbol{\xi} = \boldsymbol{\xi}^\circ$ /2/.

As a point of libration Λ_2 depends only on the squares of $\xi_i\ (i = 1, 2, 3)$. In accordance with /2/,

$$\Lambda_{\xi_1\xi_1} = 2\rho(\varphi_2 + \varphi_3 + \varphi),\ \ \Lambda_{\xi_2\xi_2} = -2\rho\varphi_2,\ \ \Lambda_{\xi_3\xi_3} = -2\rho\varphi_3$$

$$\varphi_i = F_i(\boldsymbol{\xi}^\circ)\ (i = 1, 2, 3),\ \ \varphi = -2\gamma[(\xi_1^\circ)^2/\alpha_1]/(\alpha_1\alpha_2\alpha_3)^{1/2}$$

From equilibrium condition (1.4) we have $2\rho = 1/\varphi_1$. In accordance with /2/, these expressions can be used if the function γ is assumed to be continuous on the ellipsoidal layer which covers the point of libration.

The form of lowest order in the expansions of H is

$$H_2(\mathbf{q}, \mathbf{p}) = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}p_3^2 + p_1 q_2 - p_2 q_1 - \frac{qg + h}{2}q_1^2 + \frac{g}{2}q_2^2 + \frac{\varphi_3}{2\varphi_1}q_3^2$$

and the characteristic equation of the first approximation system is

$$[\lambda^4 + (2 - h)\lambda^2 + (1 + g + h)(1 - g)](\lambda^2 + \varphi_3/\varphi_1) = 0$$

$$(g = \varphi_2/\varphi_1,\ h = (\varphi_3 + \varphi)/\varphi_1)$$

With respect to space coordinate q_3, the star performs normal oscillations with frequency $(\varphi_3/\varphi_1)^{1/2}$. Hence it suffices to study the stability in the first approximation for motions in the equatorial plane.

The equations of the first approximation of plane motion $\mathbf{z}^{\boldsymbol{\cdot}} = IG_{\mathbf{z}}$, where $\mathbf{z} = (q_1, q_2, p_1, p_2)$ and I is a simplectic matrix of fourth order, have the Hamiltonian $G(\mathbf{z}) = H_2(q_1, q_2, 0, p_1, p_2, 0)$. In the sapce of parameters h and g, the set S of points for which the necessary conditions of stability of the linear system hold, is obtained from the condition for the real part of the roots of the characteristic equation to vanish. It is given by the inequalities (Fig.1)

$$g > 0,\ h \leqslant 2,\ (g + h/2)^2 - 2h \geqslant 0,\ (1 + g + h)(1 - g) \geqslant 0$$

As distinct from the case of an external point of libration of a homogeneous ellipsoid

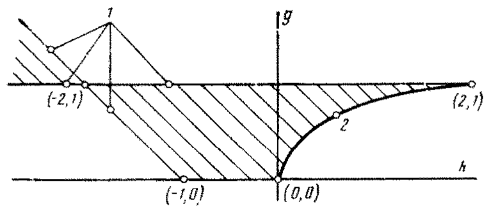

Fig.1

/1/, here the constraint $h > 0$ is not present. In the inhomogeneous case, h may be influenced by the value of the density at the point itself.

The stability conditions of the linear system become sufficient if the matrix IG_{zz} is reduced to diagonal form by a suitable linear substitution. In particular, this is true in the domain $\operatorname{Int} S$, where the roots of the characteristic equation are pure imaginary, distinct, and not equal to zero.

The set $\operatorname{Rs}(1, S) = \{(h, g): (1 + g + h)(1 - g) = 0,\ h \leqslant 2,\ g > 0\} \subset \partial S$ (curve *1*) corresponds to first-order resonance (one of the frequencies is zero). A check shows that, for all $(h, g) \in \operatorname{Rs}(1, S)$, with the exception of $(h, g) = (-2, 1)$, the elementary divisor corresponding to the zero eigenvalue of the matrix IG_{zz} is not simple. Hence, for $(h, g) = (-2, 1)$, the equilibrium in the linear approximation is stable, while for $(h, g) \in \operatorname{Rs}(1, S) \setminus \{(-2, 1)\}$ it is unstable.

It remains to consider the set $\operatorname{Rs}(2, S) = \{(h, g): (g + h/2)^2 - 2h = 0, 0 < h < 2\}$ (curve *2*) of second-order resonance. It can be shown that in this case the minor M_{14} of the matrix $JG_{zz} - i(1 - h/2)E$ (obtained by striking out the first row and fourth column) is zero only when $h = 0$ and $h = 2$. Hence, with $(h, g) \in \operatorname{Rs}(2, S)$, the eigenvalue $i(1 - h/2)$ has a non-simple elementary divisor. Hence the instability follows.

To sum up, stability is ensured in the first approximation only when $(h, g) \in \operatorname{Int} S \cup \{(-2, 1)\}$. Notice also that, with $(h, g) \in \operatorname{Int} S \cap \{(h, g): h < -2\} = S_0$, the function $G(\mathbf{z})$ is positive definite by Sylvester's criterion. Hence with $(h, g) \in S_0$, the function $H_2(\mathbf{q}, \mathbf{p})$ is also positive definite, and since $H = H_2 + \ldots$ is the integral of the exact non-linear system, then S_0 corresponds, by Lyapunov's celebrated theorem, to its stable equilibrium position.

3. Discussion. From the point of view of stellar system dynamics, a case of special interest is that when the function γ is decreasing and reaches a maximum at zero (the centre of the galaxy). It was remarked in /3/ that $A_1 + A_2 + A_3 = 2/(\alpha_1\alpha_2\alpha_3)^{1/2}$. In the case of a non-increasing function $\gamma\colon \mathbf{R}_+ \to \mathbf{R}_+$, which occurs for observed elliptic galaxies, $\varphi_i \geqslant \gamma[(\xi_1^\circ)^2/\alpha_1] A_i$ $(i = 1, 2, 3)$, where we have the exact equality when the equilibrium position is not unique and the points of libration fill the interval $[0, \xi_1^\circ]$ of the ξ_1 axis, while the density is constant on the appropriate ellipsoids.

Consequently, when the density does not increase on leaving the centre, we have the inequality $\varphi_1 + \varphi_2 + \varphi_3 \geqslant 2\gamma[(\xi_1^\circ)^2/\alpha_1]/(\alpha_1\alpha_2\alpha_3)^{1/2} = -\varphi$, which is equivalent to $1 + g + h \geqslant 0$ (Fig.1). Hence points of the domain S_0 cannot correspond to a decreasing function γ.

It is clear from this that stability only occurs in this case when $g < 1$, i.e., when $\alpha_1 < \alpha_2$. The conclusions of /1, 3/ extend to the inhomogeneous case: if an isolated point of libration is stable, it is located on the continuation of the lesser equatorial semi-axis.

After studying the stability in the first approximation, we can perform a non-linear analysis for the set of parameters $S \cap \{(h, g): 1 + g + h > 0\}$, in the same way as was done in /4/. Clearly, in plane motion we have stability everywhere in $\operatorname{Int} S \setminus S_0$, with the possible exception of some resonance curves (the Arnold-Moser theorem). With respect to spatial motions we can speak in this case of stability for the majority of initial conditions.

An unisolated equilibrium position with $g \neq 1$ $(\alpha_1 \neq \alpha_2)$ is realized only when $1 + g + h = 0$. One characteristic exponent is zero. The elementary divisor corresponding to it is not simple. The instability consits of a systematic shift along the ξ_1 axis at a constant rate which increases linearly as the distance from this axis increases. The explicit solution is easily obtained by considering the motion of the star within a homogeneous ellipsoid, in the same way as in /5/. The homogeneity follows from Theorem 1 and the fact of non-isolation. The ellipsoids considered in /5/ were those in which the positions of relative equilibrium fill an interval of the ξ_1 axis. Notice that, in the case when $1 + g + h = 0$, the Hamilton system is linear: $H(\mathbf{q}, \mathbf{p}) \equiv H_2(\mathbf{q}, \mathbf{p})$.

If $g = 1$, the points of libration are likewise not isolated, but they fill a circle in the plane of the equator. The galaxy has the form of a spheroid. It was pointed out in /4/ that here also, in the general case, we have instability for the non-linear system, due to the systematic shift along a longitude.

The author thanks V.G. Demin for suggesting the problem and for his interest.

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Translated by D.E.B.

PMM U.S.S.R., Vol.51, No.1, pp.5-9, 1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

THE DYNAMICS OF A RIGID BODY UNDER IMPACT*

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The motion of an absolutely rigid body under impact (impulsive motion) is considered. The analogy between this motion and the motion of a rigid body in a fluid is pointed out: the influence of the inertial properties of the body on the motion is defined in both cases by three second-order surfaces. The role of these surfaces when a body moves in infinite fluid was established by Zhukovskii /1/. Using the moments of the impact pulses at the point of contact (the influence of rolling friction and revolving friction), the necessary conditions are obtained for the appearance of "tangential" impact (TI). A well-known characteristic of TI is that the reaction assists in increasing the approach velocity of the points of contact of the colliding bodies, including the case when the initial approach velocity is zero ("collision without impact"). Previously /2-4/ studies of TI have only taken account of the impact pulse (the normal component of the reaction and sliding friction). The physical meaning of TI has been elucidated in a discussion of the "paradoxes" of dry friction, see /5, Appendix 2/, and in the popular literature /6/. In working devices TI often makes its appearance as dynamic selfbraking, and as unwanted cases of "sticking" and "seizing."

We consider the motion of a rigid body which belongs to a system with ideal constraining links, linear with respect to the velocities (holonomic or non-holonomic). Let the impact action on the body be specified as a principal vector $\mathbf{S}$ and principal momentum $\mathbf{L}$ of the impact force momenta, reduced to some centre. As usual in the case of impact, we neglect displacements of the material particles of the system. As the basic coordinate system we take a fixed system whose origin coincides with the centre of reduction of the impact force momenta. Determination of the motion of the rigid body under impact amounts to finding the angular velocity $\boldsymbol{\omega}$ of the body and the velocity $\mathbf{v}$ of some pole.

As the pole we take the point O of the body which coincides with the origin of the basic coordinate system (the position of the pole remains fixed during the impact, while its velocity varies from some value $\mathbf{v}^-$ before impact). We write the equations of the impulsive motion of the body (motion under impact) /7/

$$v_i - v_i^- = \frac{\partial\Phi_1}{\partial S_i} + \frac{\partial\Phi_3}{\partial S_i}, \quad \omega_i - \omega_i^- = \frac{\partial\Phi_2}{\partial L_i} + \frac{\partial\Phi_3}{\partial L_i} \tag{1}$$

$$2\Phi_1 = S^T\alpha S,\ 2\Phi_2 = L^T\beta L,\ \Phi_3 = S^T\gamma L$$

$$\alpha = \|\alpha_{ij}\|,\ \beta = \|\beta_{ij}\|,\ \gamma = \|\gamma_{ij}\|\ (i, j = 1, 2, 3)$$

Here, v_i, ω_i, S_i, L_i $(i = 1, 2, 3)$ are the projections of the vectors $\mathbf{v}, \boldsymbol{\omega}, \mathbf{S}, \mathbf{L}$ on the axes of the fixed coordinate system. The coefficients of the matrices α, β, γ are found by means of expressions for the kinetic energy Θ of the reduced system (a quadratic form in the kinetic

*Prikl.Matem.Mekhan., 51, 1, 9-14, 1987